Plane Wave Limits and T-Duality

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Abstract

The Penrose limit is generalized to show that, any leading order solution of the low-energy field equations in any one of the five string theories has a plane wave solution as a limit. This limiting procedure takes into account all the massless fields that may arise and commutes with the T-duality so that any dual solution has again a plane wave limit. The scaling rules used in the limit are unique and stem from the scaling property of the D = 11 supergravity action. Although the leading order dual solutions need not be exact or supersymmetric, their plane wave limits always preserve some portion of the Poincaré supersymmetry and solve the relevant field equations in all powers of the string tension parameter. Further properties of the limiting procedure are discussed.

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In general relativity there is a remarkably simple argument, due to Penrose [1], which shows that any spacetime has a plane wave as a limit. This universal property of the plane wave spacetimes can be proven in two steps. For this purpose one first chooses, on an arbitrary Lorentzian spacetime, a coordinate gauge in a neighborhood of a null geodesic. The coordinate patch chosen in this manner turns out to be governed by the conjugate points of the null geodesic. One then utilizes the homogeneity property of the Einstein-Hilbert action under the constant scalings of the metric and blows up the neighborhood of the geodesic through the limit. The endpoint of the procedure is always a plane wave spacetime which satisfies the Einstein equations if the initial spacetime does.

In this paper we wish to study how the Penrose limit is generalized in string theories and how it behaves under the T-duality. We shall work within the framework of the low-energy effective field theories and take into account all possible massless bosonic fields that may arise, including both the Neveu-Schwarz (NS-NS) and the Ramond (R-R) sectors as well as the Yang-Mills (YM) fields. It will be seen that the two inputs of the Penrose limit both have natural generalizations in string theories. The gauge choice employed in the limit must be generalized to incorporate the antisymmetric tensor fields and this can be done in a unified manner, applicable to all string theories . Secondly, the different scaling behaviors that must be imposed on the massless fields to produce the plane waves turn out to be rooted in the scaling property of the D=11 supergravity action. Assuming that the D=10 spacetime possesses a spacelike isometry without fixed points, but otherwise is arbitrary, we next consider the effect of T-duality on the conjugate points. We find that the conjugate points are left invariant by the duality transformations. From this observation and the invariance of the gauge conditions used, it follows that T-duality commutes with the Penrose limit. Therefore, starting from an arbitrary, leading order solution and its T-dual in any one of the five string theories, one gets in the limit a plane wave solution together with its dual plane wave in the dual theory. The fact that the plane wave solutions of type I and heterotic string theories can be obtained through the Penrose limit was noted in [2] and a related procedure was employed in [3] to generate new, exact solutions possesing only the NS-NS fields. Within the NS-NS sector, it is also known that plane waves constitute a T- duality invariant family when the isometries that correspond to the translations along the wave fronts are gauged [4]. Some other interesting aspects of the string theory plane waves can be found in [5]-[10] and certain plane wave solutions of the Type II theories are reported in [11], [12].

It is well known that the leading order terms of the low-energy Lagrangians of the type II string theories are of the from

$$\mathcal{L} = \mathcal{L}_{NS} + \mathcal{L}_R + \mathcal{L}_{CS}, \tag{1}$$

where the NS-NS sectors are described by the ten-form

$$\mathcal{L}_{NS} = \frac{1}{2\kappa_{10}^2} e^{-2\phi}[-R * 1 + 4d\phi \wedge *d\phi - \frac{1}{2} H \wedge *H], \tag{2}$$

and for the R-R sector of the IIA theory one has

$$\mathcal{L}_R = \frac{1}{4\kappa_{10}^2}[F_2 \wedge *F_2 + F_4 \wedge *F_4], \tag{3}$$

whereas for the IIB theory

$$\mathcal{L}_R = -\frac{1}{4\kappa_{10}^2}[F_1 \wedge *F_1 + F_3 \wedge *F_3 + \frac{1}{2} F_5 \wedge *F_5]. \tag{4}$$

Here R is the D=10 scalar curvature in the string frame, ϕ is the dilaton, H is the NS-NS three-form: $H = dB$ and a subscript on a R-R field denotes the degree of that form. Our spacetime conventions and the Hodge dual * are described in the appendix. The IIA R-R field strengths are even degree forms which are defined in terms of the odd degree potentials A_p by

$$F_2 = dA_1, \qquad F_4 = dA_3 + A_1 \wedge H, \tag{5}$$

whereas in the IIB theory the even degree potentials A_p give rise to the odd degree R-R field strengths:

$$\begin{aligned} F_1 = dA_0, \qquad & F_3 = dA_2 + B \wedge dA_0, \\ F_5 = dA_4 - \frac{1}{2}A_2 \wedge dB + & \frac{1}{2}B \wedge dA_2 + \frac{1}{2}B \wedge B \wedge dA_0. \end{aligned} \tag{6}$$

The Chern-Simons terms $\mathcal{L}_{CS}$ do not affect the present discussion and are ignored. In the IIB theory the field equations that follow from (1) are in harmony with the self-duality of the five-form field strength:

$$F_5 = *F_5, \tag{7}$$

but (7) must be imposed as an additional field equation. For the IIA theory (1) can be completely derived from the bosonic sector of the D=11 supergravity Lagrangian

$$\hat{\mathcal{L}} = \frac{1}{2\kappa_{11}^2}[\hat{R}\hat{*}1 - \frac{1}{2}\hat{F} \wedge \hat{*}\hat{F} - \frac{1}{6}\hat{F} \wedge \hat{F} \wedge \hat{A}], \tag{8}$$

by employing the standard Kaluza-Klein (KK) reduction. In (8) $\hat{R}$ is the scalar curvature of the D=11 metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ and $\hat{F} = d\hat{A}$ is the four-form field.

For both the type I and the heterotic strings the leading order terms of the low- energy Lagrangians can be written as $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_1$ has the same form as $\mathcal{L}_{NS}$ provided the Chern-Simons three-form of the YM field is included in the definition of H, and $\mathcal{L}_2$ stands for the YM kinetic term with the appropriate gauge group and dilaton coupling [13].

Consider now in the framework of the above Lagrangians the massless fields of any one of the five string theories. Let us first introduce a coordinate system $\{Y^+, Y^-, Y^A\}$ with $A = 1, \ldots, 8$ on the D=10 spacetime M_{10} so that the string frame metric takes the form

$$ds^2 = 2dY^+[dY^- + \alpha dY^+ + \beta_A dY^A] - C_{AB}dY^A dY^B, \tag{9}$$

where the metric functions α, β_A, C_{AB} are in general functions of all the coordinates and C_{AB} is a 8×8 positive definite symmetric matrix. Such a coordinate system can always be introduced in a neighborhood of a portion of a null geodesic provided this portion contains no conjugate points [14]. These coordinates have the propery that a null geodesic congruence is singled out in which each geodesic is given by $Y^+, Y^A = const.$ with Y^+ labelling the different geodesics and Y^- is an affine parameter along these geodesics. The coordinate system is valid as long as a conjugate point is not encountered and breaks down at the nearest conjugate point where $det(C_{AB}) = 0$.

In the NS-NS sector $(g_{\mu\nu}, \phi, B)$ one must choose a gauge not only for the D=10 metric $g_{\mu\nu}$ but also for B. Let the components of B be labelled as

$$B = B_{+-}dY^+ \wedge dY^- + B_{+A}dY^+ \wedge dY^A + B_{-A}dY^- \wedge dY^A + 1/2B_{AB}dY^A \wedge dY^B. \tag{10}$$

Then the appropriate gauge condition for B turns out to be

$$B_{-A} = 0, \tag{11}$$

and this can always be imposed in the chosen neighborhood by using the gauge freedom $B \to B + d\chi$ with a suitable one-form χ. Each potential A_p in a R-R sector also enjoys a similar gauge freedom: $A_p \to A_p + d\Lambda_{p-1} + \ldots$, involving a $(p-1)$-form Λ_{p-1} and possible compensators that depend on χ or H and whenever a R-R sector is present, one must also arrange the gauge so that

$$A^{(p)}{}_{-B_1 \ldots B_{p-1}} = 0, \tag{12}$$

holds for each potential. Here we are denoting the components of A_p by $A^{(p)}_{\mu_1 \ldots \mu_p}$ in an expansion similar to (10) and $p \geq 1$. In the cases where a YM field must be taken into account, the same applies to its Lie algebra-valued potential one-form $\mathcal{A}$:

$$\mathcal{A}_- = 0, \tag{13}$$

which means that we are working in the YM gauge: $\mathcal{A} = \mathcal{A}_+ dY^+ + \mathcal{A}_B dY^B$. This completes the first step of limiting procedure because all the gauges are now appropriately chosen. Notice that no restriction is made on the dependence of the fields on the coordinates.

The second step of the procedure starts by rescaling the coordinates chosen on the neighborhood. Let $\Omega > 0$ be a real number and introduce $\{U, V, X^A\}$ satisfying

$$Y^- = U, \qquad Y^+ = \Omega^2 V, \qquad Y^A = \Omega X^A. \tag{14}$$

When the coordinate basis one-forms are written in terms of dU, dV, dX^A in (9) and (10) but the components are not transformed, this rescaling gives us a one-parameter family of fields $(g_{\mu\nu}(\Omega), \phi(\Omega), B(\Omega))$ and the same applies to the R-R as well as the YM fields that are present. It is useful to view these as fields on a one-parameter family of spacetimes $M_{10}(\Omega)$. This allows one to interpret Ω as a scalar field on an associated D=11 manifold possessing a degenerate metric and a boundary [15]. The boundary of the D=11 manifold is located at $\Omega = 0$ which is the limit of interest. Before approaching this boundary let us introduce on $M_{10}(\Omega)$ new fields that are distinguished by overbars and are related to the old ones by

$$\bar{g}_{\mu\nu}(\Omega) = \Omega^{-2} g_{\mu\nu}(\Omega), \tag{15}$$

$$\bar{\phi}(\Omega) = \phi(\Omega), \tag{16}$$

$$\bar{B}(\Omega) = \Omega^{-2} B(\Omega). \tag{17}$$

For the R-R and the YM fields the scaling rules are

$$\bar{A}_p(\Omega) = \Omega^{-p} A_p(\Omega), \tag{18}$$

$$\bar{\mathcal{A}}(\Omega) = \Omega^{-1} \mathcal{A}(\Omega), \tag{19}$$

so that each potential is scaled according to its form degree. Allowing now $\Omega \to 0$, the overbarred fields become in the limit:

$$ds^2 = 2dUdV - C_{AB}(U) dX^A dX^B, \tag{20}$$

$$\phi = \phi(U), \tag{21}$$

$$B = \frac{1}{2} B_{KL}(U) dX^K \wedge dX^L + gauge, \tag{22}$$

where we have dropped the overbars for notational convenience. Notice that when $\Omega \to 0$ all the functions appearing in each field depend only on the coordinate U as a consequence of (14). This is, of course, valid also for the components of the R-R and YM fields. According to (5),(6) and (17),(18) a (p+1)-form R-R field strength is scaled as

$$\bar{F}_{(p+1)} = \Omega^{-p} F_{(p+1)}, \tag{23}$$

and in the limit the scaled fields take the forms

$$A_p = \frac{1}{p!} A_{K_1 \dots K_p}(U) dX^{K_1} \wedge \dots \wedge dX^{K_p} + gauge, \tag{24}$$

$$F_{(p+1)} = \frac{1}{p!} F_{-A_1 \dots A_p}(U) dU \wedge dX^{A_1} \wedge \dots \wedge dX^{A_p}. \tag{25}$$

Denoting the YM field strength by $\mathcal{F}$, one also gets in the same limit

$$\mathcal{A} = \mathcal{A}_K(U) dX^K + gauge, \qquad \mathcal{F} = \mathcal{F}_{-K}(U) dU \wedge dX^K. \tag{26}$$

What have been obtained by this limiting procedure are the general representations of the plane wave fields in the Rosen coordinates. These coordinates have the virtue of displaying the isometries but become singular at $det(C_{AB}(U)) = 0$. This can be remedied [16] by transforming all the fields to the harmonic coordinates $\{u, v, x^A\}$:

$$U = u, \qquad V = v - \frac{1}{4} \dot{C}_{AB}(U) Q^A{}_K(U) Q^B{}_L(U) x^K x^L, \qquad X^A = Q^A{}_B(U) x^B, \tag{27}$$

which covers the whole of the plane wave manifold. Here and in the sequel a dot over a quantity denotes differentation with respect to its argument. The matrix $Q^A{}_B$ is such that

$$C_{KL} Q^K{}_A Q^L{}_B = \delta_{AB}, \qquad C_{KL}[\dot{Q}^K_A Q^L{}_B - Q^K{}_A \dot{Q}^L_B] = 0, \tag{28}$$

where δ_{AB} is the D=8 Kronecker symbol. Defining an 8×8 martix $h_{AB}(u)$ by

$$h_{AB} = -[\dot{C}_{KL} \dot{Q}^L_B + C_{KL} \ddot{Q}^L_B] Q^K{}_A \tag{29}$$

the spacetime line element (20) takes the standard form

$$ds^2 = 2dudv - h_{AB}(u) x^A x^B du^2 - \delta_{AB} dx^A dx^B. \tag{30}$$

In the harmonic coordinates the field strengths (but not the potentials) retain their forms:

$$\phi = \phi(u), \qquad H = \frac{1}{2} H_{uAB}(u) du \wedge dx^A \wedge dx^B, \tag{31}$$

$$F_{(p+1)} = \frac{1}{p!} F_{uA_1 \dots A_p}(u) du \wedge dx^{A_1} \wedge \dots \wedge dx^{A_p}, \tag{32}$$

$$\mathcal{F} = \mathcal{F}_{uA}(u) du \wedge dx^A. \tag{33}$$

It can be checked that (15)-(19) which led us to the plane waves are the unique scaling rules that produce finite, non-zero field strengts. Remarkably, these rules also ensure that the D=10 Lagrangians transform homogeneously:

$$\bar{\mathcal{L}}(\Omega) = \Omega^{-8}\mathcal{L}(\Omega), \tag{34}$$

and it is possible to absorb Ω into the definition of the coupling constant: $\kappa_{10}^2 = \Omega^8\overline{\kappa}_{10}^2$. A similar behavior is encountered in the D=2 σ-model Lagrangian $\mathcal{L}_\sigma$ for the NS-NS fields. As was noted in [3] for a class of fields , $\bar{\mathcal{L}}_\sigma(\bar{\alpha'}) = \mathcal{L}_\sigma(\alpha')$ if one defines $\alpha' = \Omega^2\bar{\alpha}'$ where α' is the string tension parameter. When viewed from the D=11 supergravity framework for the IIA teory, one can see that (15)-(19) are precisely the D=10 consequences of the well known scaling behavior of (8):

$$\bar{\hat{\mathcal{L}}} = \Omega^{-9}\hat{\mathcal{L}}, \tag{35}$$

under the transformations

$$\bar{\hat{g}}_{\hat{\mu}\hat{\nu}} = \Omega^{-2}\hat{g}_{\hat{\mu}\hat{\nu}}, \qquad \bar{\hat{A}} = \Omega^{-3}\hat{A}, \tag{36}$$

and as we shall see below, the IIB theory scaling rules can then be deduced via T-duality.

An important consequence of (34) is that, if the fields were chosen intially to satisfy the relevant field equations, then (30)-(33) will again be a solution of the same equations after the limit. In all such cases all the field equations but one will be trivially satisfied by the plane waves and the remaining equation will always be a condition on the trace of $h_{AB}(u)$, relating it to the other field strengths. For the heterotic strings its precise form can be found in [2] and for IIA and IIB theories this equation will be displayed after we consider the T-duality.

We have thus seen that any leading order solution in any of the five string theories goes over to a plane wave solution in the limit. The plane wave family itself is closed under this procedure because, the limit of a plane wave is always a plane wave [15]. The limiting procedure, of course, makes no reference to a symmetry of the spacetime and (9) need not have any Killing vectors. Suppose now we assume that the initial M_{10} admits a spacelike Killing vector K^μ which has no fixed points. In such a situation one would like to know whether the Penrose limit can be tried simultaneously on a given solution and its T-dual and whether the limit of the dual solution is also a plane wave. Since T-duality can even lead to a topology change, it is not clear from the outset that the limit can be applied also to a dual solution to get another plane wave. To proceed further one needs to understand the dual patch and see whether the same type of gauges can be implemented on the dual fields. If the gauge conditions are preserved, then clearly duality will commute with the Penrose limit and the dual of a plane wave will always be a plane wave.

Therefore, let us start by considering the NS-NS sector and assume that the action of K^μ on the fields is specified in the standard manner [17], [18]. The T-duality transformations of the NS-NS fields [19] take a simple form when the fields are decomposed relative to K^μ in a KK fashion [20]. Let us introduce $\lambda^2 = -K_\mu K^\mu$ and denote by y the Killing coordinate: $Y^A = \{Y^j, y\}$, $j = 1, \ldots, 7$. Then the metric (9) can be decomposed as

$$ds^2_{(10)} = ds^2 - \lambda^2(dy + \omega)^2, \tag{37}$$

where ds^2 is the D=9 KK metric and $\omega = \omega_+ dY^+ + \omega_j dY^j$ is the twist potential. Notice that due to our gauge choice in (9), $\omega_- = 0$ and therefore, the twist potential obeys a

gauge condition which is in perfect harmony with (11)-(13). We next define the one-form $b = B_{y+}dY^+ + B_{yj}dY^j$ and write

$$B^{(10)} = B + dy \wedge b. \tag{38}$$

From now on, whenever there is a need to distinguish the D=10 fields from their D=9 descendants, our labelling of the D = 10 fields will be as in (37) or (38) with either a subscript or a superscript. In terms of the D=9 fields defined above it can be easily seen that T-duality leaves the KK metric invariant and acts on the remaining NS-NS fields as

$$\tilde{\lambda} = \lambda^{-1}, \qquad \tilde{\omega} = b, \qquad \tilde{b} = \omega, \qquad \tilde{B} = B + b \wedge \omega, \qquad \tilde{\phi} = \phi - \ln \lambda, \tag{39}$$

where tilde denotes the T-dual of a field.

Using these transformation rules and keeping in mind that the dual manifold $\tilde{M}_{10}$ possesses a new Killing coordinate $\tilde{y} : \tilde{Y}^A = \{Y^j, \tilde{y}\}$, it is useful to note first that, in string theory, a gauge choice for the axion potential must necessarily accompany the gauge choice for the metric. This is forced upon us by T-duality because, unless the gauge for $B^{(10)}$ is chosen as in (11), the dual D=10 metric does not have the form of (9) which is suitable for the limit. More precisely, among the components of (11) it is the vanishing of $B_{y-} \equiv b_-$ which ensures the form invariance of the dual metric. Since the dual metric has the same form in our gauge, it remains to see how the dual patch is related to the original one. This is also necessary because, although the KK metric is left invariant, duality maps C_{AB} to a new matrix $\tilde{C}_{AB}$ that contains contributions of ω and b and the locations of the dual conjugate points may have changed. However, one finds that

$$det(\tilde{C}_{AB}) = \lambda^{-4} det(C_{AB}), \tag{40}$$

and consequently, both patches have the same conjugate points. Notice that (40) is equivalent to the invariance of $det(C_{AB})$ relative to the Einstein frame whose metric is $g^E_{\mu\nu} = e^{-\phi/2} g_{\mu\nu}$.

Because $\omega_- = 0$, we now see that the gauge conditions in the NS-NS sector are preserved by the duality transformations. This property turns out to be universal for all the massless fields that may be present. Consider, for example, the YM field whose T-duality transformation has been studied in various contexts [21] - [23]. For definiteness let us set

$$\mathcal{A}^{(10)} = \mathcal{A} + \lambda dy \mathcal{A}_0, \tag{41}$$

and concentrate on the mapping that one gets by gauging of the isometry of the heterotic σ-model action [21]. In this framework $\mathcal{A}$ transforms as

$$\tilde{\mathcal{A}}_0 = \mathcal{A}_0, \qquad \tilde{\mathcal{A}} = \mathcal{A} + (\lambda^{-1} b - \lambda \omega) \mathcal{A}_0, \tag{42}$$

and therefore, $\tilde{\mathcal{A}}^{(10)}$ also obeys the gauge condition (13). One can check that the same conclusion is reached when the transformation rule of [22] or [23] is considered. Hence in all these cases the inclusion of the YM Chern-Simons term in H turns out to be of no consequence for our purpose and the limit of both $\tilde{\mathcal{F}}$ and $\tilde{H}$ have again the plane wave forms.

We next cosider the R-R sector. The transformations of the R-R fields [24] can be conveniently displayed also by using the 9 +1 decomposition. Following [25] we define the D=9 fields

$$F_2{}^{(10)} = F_2 + F_1 \wedge (dy + \omega), \qquad F_4{}^{(10)} = F_4 + F_3 \wedge (dy + \omega), \tag{43}$$

for the IIA theory. For the IIB theory the correspoding decompositions are

$$F_1{}^{(10)} = F_1, \qquad F_3{}^{(10)} = F_3 + F_2 \wedge (dy + \omega), \qquad F_5{}^{(10)} = F_5 + F_4 \wedge (dy + \omega). \quad (44)$$

In terms of the D=9 quantities the T-duality rules which map the IIA theory into IIB theory are then

$$\tilde{F}_1 = -F_1, \qquad \tilde{F}_2 = F_2, \qquad \tilde{F}_3 = -F_3, \qquad \tilde{F}_4 = F_4, \quad (45)$$

together with the rule that $\tilde{F}_5$ is the D=9 dual of $\tilde{F}_4$. Since these rules only involve ω and the R-R field strengths of the IIA theory that one started with, it is obvious that the gauge choices are preseved also within the R-R sector. Notice that one can also infer the scaling rules (16)-(18) for the IIB fields from those of the IIA theory by invoking the duality (45).

It therefore follows that if one starts with a set of IIA fields and finds the Penrose limit, then the limit of the dual set of fields in the IIB theory will be simply the dual of the plane waves obtained in the IIA theory. Because the two Killing coordinates need not be the same, one needs two different D =10 harmonic coordinates to describe such a dual pair of plane waves. One can, of course, always use the same D=9 harmonic coordinates on both of the solutions. We shall display the general forms of a IIA - IIB dual pair in such coordinates. For example, without any loss of generality the metric that one obtains from (9) for the IIA theory can be brought to the form

$$ds^2 = 2dudv - h_{ij}(u)x^i x^j du^2 - \delta_{ij}dx^i dx^j - (da - \gamma du)^2, \quad (46)$$

where a is a new Killing coordinate: $K^\mu = -\lambda \delta^\mu{}_a$ and γ is a function which depends linearly on all the transverse coordinates $x^A = \{x^j, a\}$:

$$\gamma = \gamma_A(u)x^A. \quad (47)$$

Here $\gamma_A(u)$ is completely characterized by the norm and the twist of the Killing one-form $K = K_\mu dx^\mu$. Noting that $K \wedge dK = \lambda^2 K \wedge d\omega$ still holds after $\Omega \to 0$ and writing $d\omega = \dot{\omega}_j(u)du \wedge dx^j$ in the harmonic coordinates, one gets

$$\gamma_A = \{\lambda\dot{\omega}_j, \dot{\lambda}/\lambda\}. \quad (48)$$

The dual of this metric, for example, in the IIB theory is obtained simply by dualizing the Killing coordinate and γ:

$$d\tilde{s}^2 = 2dudv - h_{ij}(u)x^i x^j du^2 - \delta_{ij}dx^i dx^j - (d\tilde{a} - \tilde{\gamma}du)^2, \quad (49)$$

where

$$\tilde{\gamma}_A = \{\lambda^{-1}\dot{b}_j(u), -\dot{\lambda}/\lambda\}, \quad (50)$$

and $\dot{b}_j$ are defined by $db = \dot{b}_j(u)du \wedge dx^j$. The dual NS-NS three-forms are given by

$$H = \frac{1}{2}p_{jk}(u)du \wedge dx^j \wedge dx^k + \lambda^{-1}\dot{b}_j(u)du \wedge da \wedge dx^j, \quad (51)$$

$$\tilde{H} = \frac{1}{2}p_{jk}(u)du \wedge dx^j \wedge dx^k + \lambda\dot{\omega}_j(u)du \wedge d\tilde{a} \wedge dx^j, \quad (52)$$

where $p_{jk}(u)$ are arbitrary functions. The dilaton ϕ is again an arbitrary function of u and $\tilde{\phi} = \phi - \ln\lambda$.

The R-R field strengths also have a similar structure. Provided $p \geq 2$, a D = 10 R-R p-form field strength has the form

$$F_p = \frac{1}{(p-1)!} f^{(p)}{}_{j_1 \dots j_{p-1}} du \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{p-1}} + \frac{1}{(p-2)!} k^{(p)}{}_{j_1 \dots j_{p-2}} du \wedge da \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{p-2}}, \tag{53}$$

with arbitrary amplitudes $f^{(p)}_{j_1 \dots j_{p-1}}(u)$ and $k^{(p)}_{j_1 \dots j_{p-2}}(u)$, and the remaining case is simply: $F_1 = f^{(1)} du$ where $f^{(1)}(u) = \dot{A}_0(u)$. In this notation a set type IIB R-R plane wave fields will be the T-dual of its IIA counterpart if

$$\tilde{f}^{(1)}(u) = -k^{(2)}(u), \quad \tilde{k}^{(3)}_j(u) = f^{(2)}{}_j(u), \quad \tilde{f}^{(3)}_{jk}(u) = -k^{(4)}{}_{jk}(u), \quad \tilde{k}^{(5)}_{jkl}(u) = f^{(4)}{}_{jkl}(u). \tag{54}$$

In order to satisfy the self-duality condition (7), the remaining D = 9 field of the IIB theory must obey

$$\tilde{f}^{(5)}_{jklm} = \frac{1}{3!} \epsilon_{jklm}{}^{noi} \tilde{k}^{(5)}_{noi}, \tag{55}$$

which means that it is the dual of $\tilde{k}^{(5)}_{ijk}$ in the D=7 flat transverse space.

A characteristics of the plane waves is the vanishing of all the scalar invariants that one can construct from the field strengths and this is, of course, shared by all the above field strengths. Moreover, all the field strengths are both closed and co-closed under exterior differentiation. The exterior product of a field strength with another field strength or its Hodge dual is always zero. Due these properties, all the field equations of IIA or IIB theory, excepting the G_{uu} component of the Einstein equations are automatically satisfied. For example, in the IIA theory the only implication of the field equations is that

$$h_{jj} = 2\ddot{\phi} - \frac{\ddot{\lambda}}{\lambda} - \frac{\lambda^2}{2}\dot{\omega}_j\dot{\omega}_j - \frac{1}{2\lambda^2}\dot{b}_j\dot{b}_j - \frac{1}{4}p_{jk}p_{jk} - \frac{1}{2}e^{2\phi}[Ramond], \tag{56}$$

where *Ramond* denotes the contributions of the R-R sector:

$$Ramond = (k^{(2)})^2 + f^{(2)}{}_j f^{(2)}{}_j + \frac{1}{2} k^{(4)}{}_{ij} k^{(4)}{}_{ij} + \frac{1}{6} f^{(4)}{}_{jkl} f^{(4)}{}_{jkl}, \tag{57}$$

and this fixes the trace of $h_{jk}(u)$ in terms of the other fields. Here the trace and the other sums refer to the metric δ_{jk} on the D=7 flat transverse space.

There are two points worth noting in (56). First, one can infer from (56) that the most general type IIA (or IIB) plane wave solution involves a total of 128 arbitrary functions. Half of these always belong to the NS-NS sector and the remaining 64 functions come from the R-R sector. These numbers are precisely the numbers of degrees of freedom of the massless states in the first quantized type II string theories. Secondly, (56) shows that, with the above assumptions about the isometry, plane waves constitute a T-duality invariant family. This result was already known [4] within the NS-NS sector where h_{jk} and p_{jk} are inert to duality and the dual roles of ϕ and λ as well as of $\dot{\omega}_j$ and $\dot{b}_j$ are manifest. Because (54) and (55) hold, the duality invariance of the R-R sector is also now manifest.

Some particular choices of the arbitrary functions appearing in (56) lead to interesting generalizations of the well known solutions. One such class is the case of the sandwich

waves [16] where all the amplitutes are taken to be non-zero only over a finite interval of u. This leads to a spacetime in which two flat regions are connected by plane waves of finite duration. It would be interesting to see how the boundary state formalism for the D-branes can be applied on such a geometry. Another class is obtained by choosing all the amplitudes to be constants so that u is also a global Killing coordinate and $h_{jk} = c\delta_{jk}$, where c is a constant determined by (56). With these assumptions one gets a generalization of the Nappi-Witten solution [7] to D=10 and to a non-zero R-R sector. In this case the NS-NS sector of the solution corresponds to a WZNW model based on the ten-dimensional Heisenberg group [26].

Notice that although a spacelike isometry is initially an additional assumption, this does not entail an additional symmetry on the spacetime that one gets through the limit. Viewed after the limit, the Killing vector K^μ is simply a member of the 17-parameter group of motions of the D= 10 plane wave spacetimes. If the orbits of K^μ are assumed to be not compact, which is the case of the usual plane waves, M_{10} as well as its dual has the standard R^{10} topology after the limit. In this case duality is just a mapping between two different sets of solutions. When K^μ has compact orbits so that the quantum equivalence of the underlying string theories can be considered, the manifold M_{10} that one obtains by the limit has the $R^9 \times S^1$ topology and moreover, dualization does not bring in a twist at the field theory level. In other words, after the limit the dual manifold has again the $R^9 \times S^1$ topology. Since the topological properties are not hereditary properties in the sense of [15], a particular topology for M_{10} need not be assumed prior to the limit in the non-compact case. In the case of a compact isometry, one must start with a $M_{10} = M_9 \times S^1$, where the topolgy of M_9 is initially unspecified but $y \equiv y + 2\pi\sqrt{\alpha'}R_b$ on S^1 . Letting $y = \Omega x$ and noting that α' also scales, this implies $x \equiv x + 2\pi\sqrt{\bar{\alpha}'}R_b$ on the plane wave Killing coordinate x. In the coordinate system of (46) this identification corresponds to a "local compactification": $a \equiv a + 2\pi\sqrt{\bar{\alpha}'}\lambda(u)R_b$.

In type I and heterotic theories plane waves are known to be exact solutions which preserve half of the Poincaré supersymmetry [2],[5], [6]. These plane waves therefore satisfy the field equations not only at the leading order but in all orders of α'. Moreover, their behavior under the T-duality is not affected by the higher order α' corrections to the Buscher rules [21]. When the self-dual five-form is switched off, type IIB plane waves are also exact solutions [12]. The plane waves of the IIA theory, on the other hand, are known to admit at least chiral Killing spinors which preserve again 1/2 supersymmetry [11]. These Killing spinors do not depend on the Killing coordinate used in duality and consequently, one can conclude that the plane wave duals in the IIB theory are also supersymmetric [9]. Since the solutions on which the limit is applied are not necessarily exact, supersymmetric or have exact T-duals, it is clear that the absence of these basic properties are not hereditary. The presence of any one of these properties in an initial configuration is, of course, hereditary.

Appendix

Our conventions are as follows: In all $D \geq 2$ we use the "mostly minus" signature $(+, -, \ldots, -)$ and the orientation $\epsilon_{012\ldots D-1} = 1$. The Ricci tensor is defined as $R_{\mu\nu} = R^\lambda{}_{\mu\nu\lambda}$ and the Riemann curvature obeys $(\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu)T_\kappa = R^\lambda{}_{\kappa\mu\nu}T_\lambda$ for an arbitrary

T_μ. The Hodge dual of a p-form ($p \leq D$) is defined by

$$* \left(V^{\alpha_1} \wedge \ldots \wedge V^{\alpha_p}\right) = \frac{(-1)^{(D-1)}}{(D-p)!} \epsilon^{\alpha_1 \ldots \alpha_p \alpha_{p+1} \ldots \alpha_D} V_{\alpha_{p+1}} \wedge \ldots \wedge V_{\alpha_D},$$

in terms of an orthonormal basis $\{V^\alpha\}$.

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